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Letter to the Editor

On Best Simultaneous Approximation

The purpose of this note is to communicate the statement of two theoren s which are somewhat of a generalization of a result of Diaz and McLaughlı. [1]. (Details will appear elsewhere.)

Let X be a normed linear space and K a subset of X.

DEFINITION. Given any bounded subset $F \subset X$, define

$$d(F, K) = \inf_{k \in K} \sup_{f \in F} \| f - k \|. \tag{1}$$

An element $k^* \in K$ is said to be a best simultaneous approximation to the set F, if

$$d(F, K) = \sup_{f \in F} \| f - k^* \|. \tag{2}$$

Diaz and McLaughlin [1] and Dunham [2] have considered the problem of simultaneous approximation of the following case: $X = C[a, b]$, K a non-empty subset of X and $F = \{f_1, f_2\}$. Goel, Holland, Nasim, and Sahney [3] studied the problem of X a normed linear space, K a subset and $F = \{f_1, f_2\}$.

Using the same procedure as in [3], it is possible to study the problem where

$$F = \{f_1, f_2, \ldots, f_n\}. \tag{3}$$

The aim here is to report the results on the same problem, where F is any compact subset of X.

The following are the main theorems.

THEOREM 1. *Let K be a finite-dimensional subspace of a strictly convex normed linear space X. Then there exists one and only one best simultaneous approximation from the elements of K to any given compact subset $F \subset X$.*

THEOREM 2. *Let K be a closed and convex subset of a uniformly convex Banach space X. For any compact subset $F \subset X$, there exists a unique best approximation to F from the elements of K.*

The proofs can be developed with the aid of the following lemmas:

LEMMA 1. *Let $k \in X$ and F be a bounded subset of X. Then*

$$\phi(k) \equiv \sup_{f \in F} \| f - k \| \tag{4}$$

is a continuous functional on X.

LEMMA 2. *If K is a finite-dimensional subspace of a normed linear space X, then there exists a best simultaneous approximation $k^* \in K$ to any given compact subset $F \subset X$.*

LEMMA 3. *Let K be a convex subset of X, and $F \subset X$. If k_1 and $k_2 \in K$ are best simultaneous approximations to F by elements of K then*

$$\bar{k} = \lambda k_1 + (1 - \lambda) k_2, \qquad 0 \leqslant \lambda \leqslant 1 \tag{5}$$

is also a best simultaneous approximation to F.

Remark. It has been pointed out by the referee that Dunham has included Theorem 1 in one of his papers which has been recently refereed.

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